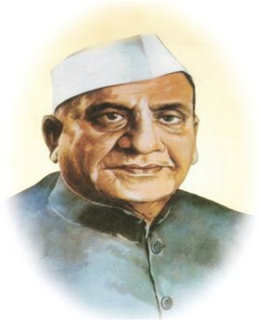

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Topic: Linear Transformation

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Let U and V be two vector spaces over the same field F.

A function $T : U \rightarrow V$ is said to be linear transformation from U to V if

i) $T(u + v) = T(u) + T(v) \quad \forall \ u, v \in U$

ii) $T(\alpha u) = \alpha T(u) \quad \forall \ u \in U, \ \alpha \in F$

In other words a function $T : U \rightarrow V$ is said to be linear transformation from U to V which associates to each element $u \in U$ to a unique element $T(u) \in V$ such that

$T(\alpha u + \beta v) = \alpha T(u) + \beta T(v) \quad \forall \ u, v \in U$ and $\alpha, \beta \in F$

Properties of linear Transformation

If $T : U \to V$ is a linear transformation from U to V, then

i) . $T(0) = 0'$, where $0 \in U$ and $0' \in V$

We have $T(\alpha u) = \alpha T(u)$ $\quad \forall\ u \in U,\ \alpha \in F$

Put $\alpha = 0 \in F$, then $T(0u) = 0T(u) = 0'$

$\therefore\ T(0) = 0'$

ii) Again we have $T(\alpha u) = \alpha T(u)$ $\quad \forall\ u \in U,\ \alpha \in F$

put $\alpha = -1 \in F$, then $T(-1.u) = -1. = -T(u)$

$\therefore\ T(-1.u) = -T(u)$

iii) $T(\alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 u_3 + \ldots + \alpha_n u_n) = T(\alpha_1 u_1) + T(\alpha_2 u_2 + \alpha_3 u_3 + \ldots + \alpha_n u_n)$

$= \alpha_1 T(u_1) + T(\alpha_2 u_2) + T(\alpha_3 u_3 + \ldots + \alpha_n u_n)$

$= \alpha_1 T(u_1) + \alpha_2 T(u_2) + T(\alpha_3 u_3 + \ldots + \alpha_n u_n)$

....

$= T(\alpha_1 u_1) + T(\alpha_2 u_2) + T(\alpha_3 u_3) + \ldots + T(\alpha_n u_n)$

iv) $T(u - v) = T(u) - T(v)$ $\quad \forall\ u, v \in U,$

Now $T(u - v) = T\{u + (-v)\}$

$= T(u) + T(-v) = T(u) - T(v)$ $\quad (\ T(-v) = -T(v))$

$\therefore\ T(u - v) = T(u) - T(v)$

Example : The function T: $R^2 \rightarrow R^2$, defined by (x, y) = (x + 1, y + 3) is not a Linear Transformation.

Solution: Consider (x, y) = (1, 1) and show that T(α(1, 1) = αT(1, 1).

Let T: $R^2 \rightarrow R^2$ be defined by (x, y) = (x + 1, y + 3) $\forall$ (x, y) $\in R^2$

$\Rightarrow$ T(1, 1) = (2, 4)

Now T(3(1, 1) = T(3, 3) and 3T(1, 1) = 3(2, 4) = (6, 12)

Thus T(3(1, 1) $\neq$ 3T(1, 1), hence T is not linear transformation.

Example: (NET) which of the following is L.T. from R^3 to R^2……

Kernel of L.T.:

Let $T : U \to V$ be a linear transformation from U to V.

Null space or kernel of T and is defined as

Ker = { u$\in$ U | T(u) = **0** = zero vector of V} [if T(**0**) = **0** $\Rightarrow$ **0** $\in$ KerT $\subset$ U]

Range of L.T. :

Let $T : U \to V$ be a linear transformation from U to V.

Range of T is denoted by R(T) and defined as

R(T) = { T(u) | u $\in$ U} [R(T) = T(U)]

Nullity of T:

The dimension of null space is called nullity of T.

Denoted by n(T) or dimN(T).

Rank of T :

The dimension of R(T) is called rank of T.

Denoted by r(T) or Dim(R(T).

Theorem . Let $T : U \rightarrow V$ be a linear transformation from U to V. Then

(a) R(T) is a subspace of .

(b) N(T) is a subspace of .

(c) T is 1-1$\Leftrightarrow$ N(T) is a zero subspace of U

(d) $T[u_1 + u_2 + u_3 + \ldots + u_n] = R(T) = [Tu_1 + Tu_2 + Tu_3 + \ldots + Tu_n]$

(e) U is a finite dimensional vector space $\Rightarrow$ dimR(T) $\leq$ dimU.

Theorem . Let $T : U \to V$ be a linear transformation from U to V. Then

a) If T is 1-1 and u_1 , u_2 , u_3 ,… , u_n are LI vectors in U, then Tu_1 ,Tu_2 , Tu_3 ,…,Tu_n are LI vectora in V.

b) If v_1 , v_2 , v_3 ,… , v_n are LI in R(T) and u_1 , u_2 , u_3 ,… , u_n are vectors in U such that $Tu_i = v_i$ for i = 1,2,3…n. Then {u_1 , u_2 , u_3 ,… , u_n} is LI.

Theorem . Let $T : U \to V$ be a linear map and U be finitely dimensional vector space.

Then dimR(T) + dimN(T) = dim (U)

i.e, Rank + Nullity = dim. of domain.

Theorem. If U and V are same finitely dimensional vector spaces over the same field, then a linear map T: U $\to$ V is 1-1 $\Leftrightarrow$ T is onto.

Corollary: Let $T : U \to V$ be a linear map and dimU = dimV = a finite positive integer. Then following statements are equivalent:

a) T is onto b) R(T) = V c) dimR(T) = dimV

d) dim N(T0 =0 e) N(T0 =0 f) T is 1-1.

Algebra of Linear Transformations

A: Let U and V be two vector spaces over the field F.

Let T_1 and T_2 be two linear transformations from U to V.

i) Then the function $(T_1 + T_2)$ defined by

$$(T_1 + T_2)(u) = T_1(u) + T_2(u) \qquad \forall\ u \in U$$

is a linear transformations from U to V.

ii) If $\alpha \in F$ is any element, then the function (αT) defined by

$$(\alpha T)u = \alpha T(u) \qquad \forall\ u \in U$$

is a linear transformations from U to V.

[The set of all linear transformations L(U, V) from U to V, together with vector addition and scalar multiplication defined above, is a vector space over the field F.]

B: Let U be an m-dimensional and V be an n- dimensional vector spaces over the same field F.

Then the vector space L(U, V) if finite- dimensional and has dimension mn.

C: Let U, V and W be vector spaces over the field F. Let $T_1 : U \rightarrow V$ and $T_2 : V \rightarrow W$, then the composition function $T_2.T_1$ is defined by

$$T_2.T_1(u) = T_2[T_1(u)] \qquad \forall\, u \in U$$

is a linear transformations from U to W.

Linear operator : If V is a vector space over the field F, the a linear transformation from V into V is called a linear operator.

Example: Let T_1 and T_2 be two linear transformations from R^2 (R) into R^2 (R) defined by $T_1(x, y) = (x + y, 0)$ and $T_2(x, y) = (0, x - y)$, then $T_2T_1 \neq T_1T_2$.

Solution: $T_1T_2(x, y) = T_1(T_2(x, y)) = T_1(0, x - y) = (x - y, 0)$

$T_2T_1(x, y) = T_2(T_1(x, y)) = T_2(x + y, 0) = (0, x + y)$,

$\therefore \quad T_2T_1 \neq T_1T_2$.

If T is a linear operator on V, then we can compose T with T as follows

$T^2 = TT$

$T^3 = TTT$

....

$T^n = TTT....T$ (n times)

Remark: If $T \neq 0$, then we define $T^0 = I$ (identity operator)

Theorem: Let V be a vector space over field F, le T, T_1, T_2, and T_3 be linear operators on V and let α be an element in F, then

i) IT = TI = T. I being an identity operator.

ii) $T_1(T_2 + T_3) = T_1T_2 + T_1T_3$, and $(T_2 + T_3) T_1 = T_2T_1 + T_3T_1$.

iii) $T_1(T_2 \ T_3) = (T_1T_2)T_3$.

iv) $\alpha(T_1T_2) = (\alpha T_1)T_2 = T_1 (\alpha T_2)$.

v) $T\mathbf{0} = \mathbf{0}T = \mathbf{0}$, **0** being a zero linear operator.

Invertible linear transformation:

A linear transformation $T : U \to V$ is called invertible or regular if there exists a unique linear transformation $T^{-1} : V \to U$ such that $T^{-1} T = I$ is identity transformation on U and TT^{-1} is the identity transformation on V.

T is invertible $\Leftrightarrow$ i) T is 1-1 ii) T is onto i.e dimR(T) = V

Theorem: Let U and V be vector spaces over the same field F. and let $T : U \to V$ be a linear transformation, If T is invertible, then T^{-1} is a linear transformation from V into U.

Theorem: Let $T_1: U \to W$ and $T_2: V \to W$ be invertible linear transformations. Then $\mathbf{T_1T_2}$ is invertible and $\mathbf{(T_2T_1)^{-1} = T_1^{-1}T_2^{-1}}$.

Non-singular linear transformation:

Let U and V be vector spaces over the field F. Then a linear transformation $T : U \to V$ is called non-singular if T is 1-1 and onto. ($T^{-1} : V \to U$ exists)

Theorem: Let $T : U \to V$ be a non-singular linear map. Then $T^{-1} : V \to U$ is a linear 1-1 and onto.

Example: Let $T: V_3 \to V_3$ be a linear map defined by
$T(x_1, x_2, x_3) = (x_1 + x_2 + x_3, x_2 + x_3, x_3)$. Show that T is non-singular and find T^{-1}.

Solution: We have, T is non-singular = T is 1-1 and onto.

$\therefore$ First we show that T is 1-1,

Let $(x_1, x_2, x_3) \in N(T)$ $\Rightarrow T(x_1, x_2, x_3) = \mathbf{0}$

$\Rightarrow (x_1 + x_2 + x_3, x_2 + x_3, x_3) = \mathbf{0}$

$\therefore$ $x_1 + x_2 + x_3 = 0$, $x_2 + x_3 = 0$, $x_3 = 0$ $\Rightarrow$ $x_1 = 0 = x_2 = x_3$.

$\therefore$ $(0,0,0) \in N(T) \Rightarrow N(T) = \{0\} \Rightarrow$ T is 1-1

Now dimension of domain and dimension of co-domain are same i.e. t is onto.

$\therefore$ T is 1-1 and onto $\Rightarrow$ T is non-singular.

Next, to find T^{-1},

Let $T^{-1}(y_1, y_2, y_3) = x_1, x_2, x_3$.

$\Rightarrow$ $T(x_1, x_2, x_3) = (y_1, y_2, y_3)$

$\Rightarrow$ $(x_1 + x_2 + x_3, x_2 + x_3, x_3) = (y_1, y_2, y_3)$

$\therefore$ $x_1 + x_2 + x_3 = y_1$, $x_2 + x_3 = y_2$, $x_3 = y_3$. $\Rightarrow$ $x_3 = y_3$, $x_2 = y_2 - y_3$ $x_1 = y_1 - y_2$,

$\therefore$ $\mathbf{T^{-1}(y_1, y_2, y_3) = (y_1 - y_2, y_2 - y_3, y_3)}$.

Co-ordinate vector:

Let V be a finitely dimensional vector space over a field F and let dimV = n , then B = { $v_1 + v_2 + v_3 + \ldots + v_n$} is an ordered basis of V and for $v \in V$ can be uniquely written as

$v = \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 + \ldots + \alpha_n v_n$

where the scalars $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n$ are fixed for v.

The vector $(\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n)$ is called the co-ordinate vector of **v** relative to the ordered basis B and denoted by $[v]_v$.

i.e. $$[v]_B = (\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n) = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ . \\ . \\ \alpha_n \end{bmatrix}$$

Example: Let B = { (1,1,1), (1,0,1), (0,0,1)} be a for V_3. Find the co-ordinate vector (2,3,4) $\in V_{3.}$ relative to basis B.

Solution. Let B = { v_1 , v_2 ,v_3 } be an ordered basis for V_3, and v_1 , = (1,1,1), v_2 = (1,0,1), v_3 = (0,0,1), . Denote. v = (2,3,4) $\in V_3$ =L(B).

$$v = \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 \quad \alpha_i \in F$$

$$(2,3,4) = \alpha_1(1,1,1), + \alpha_2(1,0,1), + \alpha_3(0,0,1) = (\alpha_1+\alpha_2, \alpha_1, \alpha_1 + \alpha_2 + \alpha_3)$$

$(\alpha_1+\alpha_2 = 2, \ \alpha_1 = 3, \alpha_1 + \alpha_2 + \alpha_3 = 4 \Rightarrow \alpha_1 = 3, \alpha_2 = -1, \alpha_3 = 2$

$[v]_B = (\alpha_1, \alpha_2, \alpha_3) = (3, -1, 2)$ = co-ordinate vector of (2,3,4) relative to B.

Example: Let B = { (1,-1,3), (-3,4,2), (2,-2,4)} be a for V_3. Find the co-ordinate vector (8,-9,6) $\in V_3$ relative to basis B.

Matrix associated with a linear map:

Let U and V be vector spaces of dimension n and m respectively over the same field F.

Consider $B_1 = \{ u_1 + u_2 + u_3 + \ldots + u_n \}$

and $B_2 = \{ v_1 + v_2 + v_3 + \ldots + v_m \}$

are the ordered basis of vector spaces U and V respectively.

Define a linear map $T : U \to V$. where T stands the vectors of B_1 to $Tu_1, Tu_2, Tu_3, \ldots, Tu_n$ in V

Then Tu_1 = linear combination of basis vectors B_2 of V

$Tu_1 = \alpha_{11}v_1 + \alpha_{21}v_2 + \alpha_{31}v_3 + \ldots + \alpha_{m1}v_m.$

$Tu_2 = \alpha_{12}v_1 + \alpha_{22}v_2 + \alpha_{32}v_3 + \ldots + \alpha_{m2}v_m.$

$Tu_1 = \alpha_{13}v_1 + \alpha_{23}v_2 + \alpha_{33}v_3 + \ldots + \alpha_{m3}v_m.$

…… …….

$Tu_n = \alpha_{1n}v_1 + \alpha_{2n}v_2 + \alpha_{31}v_3 + \ldots + \alpha_{mn}v_m.$

$$Tu_j = \sum_{i=1}^{m} \alpha_{ij} v_i = \begin{bmatrix} \alpha_{1j} \\ \alpha_{2j} \\ \alpha_{3j} \\ . \\ . \\ \alpha mj \end{bmatrix}$$

is the co-ordinate vector with respect to the ordered basis B_2.

Each …$\alpha_{ij} \in$ F, then

$$M = \begin{bmatrix} \alpha_{11} & \alpha_{12} & . & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & . & \alpha_{2n} \\ . & . & . & . \\ . & . & . & . \\ \alpha_{m1} & \alpha_{m2} & . & \alpha_{mn} \end{bmatrix}$$

M= [matrix] is the matrix whose jth column is $\begin{bmatrix} \alpha_{1j} \\ \alpha_{2j} \\ \alpha_{3j} \\ . \\ . \\ \alpha mj \end{bmatrix}$

which is the coordinate vector relative to the basis B_2.

This matrix M is called the matrix of T or the matrix associated with the linear map T with respect to bases B_1 and B_2.It is denoted by $(T: B_1, B_2)$.

$\therefore$ $(T: B_1, B_2)$ = ...$(\alpha_{ij})_{mxn}$ = [matrix]

$$(T:B_1,B_2)=(\alpha_{ij})_{m\times n}=\begin{bmatrix} \alpha_{11} & \alpha_{12} & . & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & . & \alpha_{2n} \\ . & . & . & . \\ . & . & . & . \\ \alpha_{m1} & \alpha_{m2} & . & \alpha_{mn} \end{bmatrix}$$